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Invariant Imbedding and Generalizations of the WKB Method and the Bremmer Series*

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1. INTRODUCTION

One of the most frequently used approximation devices in wave propagation theory and quantum mechanics is that which commonly goes by the name "WKB method" (Wentzel, Kramers, Brillouin). (This is actually a misnomer, since the ideas involved go back to Liouville.) Various modifications and improvements of this method are known. Perhaps the one having the most physical significance is that due to H. Bremmer [8]. Bremmer started with the WKB approximation and used it to develop an infinite series which, under certain conditions, converges to the solution of the wave equation. A very interesting aspect of this Bremmer series is that each term has a well-defined physical meaning.

During the preparation of the manuscript for a book on the subject of invariant imbedding [6], it was noticed that the imbedding method, properly applied to the wave equation, yielded the WKB approximation and the Bremmer series. While invariant imbedding has its source in the study of transport theory, it is now divorced from its physical origins and can be viewed as a strictly mathematical device. This suggested to the author that WKB-like approximations and Bremmer-like series should exist for a wide class of equations, quite independent of any physical origins. That matter is pursued in this paper.

In Section 2, a generalized WKB approach is formulated for a relatively arbitrary two-point boundary-value problem posed for a system of two coupled linear homogeneous differential equations in two unknowns. While the language of physics is often used for clarity of presentation, the analysis is really purely mathematical and makes strong use of the reflection and transmission functions of the imbedding method.

Section 3 logically carries the technique to the development of a Bremmer-like series. Again, it is noted that each term can be considered, via the

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reflection and transmission functions, as having "physical" meaning, but once again the reasoning is independent of this interpretation.

Section 4 admittedly constitutes a digression. Here we examine some possible alternative approaches which may have occurred to the reader and indicate why each of them seems less satisfactory than the route actually taken in the preceding sections.

The matter of convergence of the new Bremmer series is faced in Section 5. The approach is strongly influenced by the work of Atkinson [1]. A convergence criterion which is, in a sense, "best possible" is obtained. In this section and in those which follow, the imbedding method plays no role. Its sole purpose is served in the derivations of the earlier sections.

The application of our ideas to the wave equation is the subject of Section 6. We show how to specialize to get the classical WKB and Bremmer results, and how to obtain the improvements due to Sluijter [11]. Actually, the section contains the ingredients for still further improvements, but they are left to the reader.

Section 7 is a potpourri of ideas which are suggested by our approach. Most of these are not pursued rigorously and may provide the interested reader with areas for further investigation.

The final section summarizes the work and also adds a few more somewhat less specific ideas for future research.

2. The Reflection and Transmission Function Approach to the WKB Method

We shall consider throughout much of this paper a system of the form

$$\begin{aligned} \frac{du}{dz} &= A(z)\,u(z) + B(z)\,v(z), \\ -\frac{dv}{dz} &= C(z)\,u(z) + D(z)\,v(z), \end{aligned} \tag{2.1a}$$

where all the functions are scalars and the coefficients $A(z)$, $B(z)$, etc. are at least piecewise continuous over some interval $Y \leqslant z \leqslant X$. We impose the boundary conditions

$$u(y) = 0, \qquad v(x) = 1, \tag{2.1b}$$

and suppose that (2.1a, b) is soluble for all x and y such that $Y \leqslant y < x \leqslant X$.

This problem may be thought of as a mathematical model of an abstract transport process (see [6, 12]). In this interpretation, $u(z)$ represents a flux of particles to the right at z, $v(z)$ a flux to the left, and conditions (2.1b) assert that no right-moving particles enter the system at the left, $z = y$, while one particle per unit time is injected to the left at $z = x$. The assumption that the problem is soluble for all y and x such that $Y \leqslant y < x \leqslant X$ is physically equivalent to the statement that any such configuration is "subcritical".

While it is convenient to keep this physical model in mind, and we shall refer to it often, it is important to recognize that (2.1a, b) need not arise in such a context. Rather, (2.1a, b) should be viewed as a quite arbitrary linear homogeneous two-point boundary-value problem. The conditions imposed on $A(z)$, $B(z)$, etc., are merely to insure that the classical existence and uniqueness theorems for solutions of (2.1a) can be applied. Obviously, these conditions are overly restrictive.

Using the basic ideas of invariant imbedding, one can associate with such a problem a pair of reflection and transmission functions. Specifically,

$$R_r(y, x) = u(x), \qquad T_r(y, x) = v(y). \tag{2.2}$$

Again this concept and indeed the notation lie in the physical background. Thus $R_r(y, x)$ is the "reflected" flux at x when the input at y is zero and that at x is unity. Similarly, $T_r(y, x)$ is the "transmitted" flux at y under the same conditions. The subscript "r" indicates that the input is at the right, $z = x$. However, the functions R_r and T_r are clearly well-defined mathematically and are quite independent of these interpretations.

Let us now turn to another set of boundary conditions.

$$u(y) = 1, \qquad v(x) = 0. \tag{2.1c}$$

If we assume that (2.1a, c) is also soluble for all y and x, $Y \leqslant y \leqslant x \leqslant X$, then we may define

$$R_l(y, x) = v(y), \qquad T_l(y, x) = u(x). \tag{2.3}$$

The physical interpretation of these reflection and transmission functions can be left to the reader.

Equations satisfied by the R and T functions have been derived in a variety of ways (see [6, 9, 10]). For our purposes, a full set is not needed. We require only

$$\frac{\partial R_r}{\partial x}(y, x) = B(x) + (A(x) + D(x))\, R_r(y, x) + C(x)\, R_r^2(y, x), \tag{2.4a}$$

$$-\frac{\partial R_l}{\partial y}(y, x) = C(y) + (A(y) + D(y))\, R_l(y, x) + B(y)\, R_l^2(y, x), \tag{2.4b}$$

$$\frac{\partial T_r}{\partial y}(y, x) = (D(y) + B(y)\, R_r(y, x))\, T_r(y, x), \tag{2.4c}$$

$$-\frac{\partial T_l}{\partial y}(y, x) = (A(y) + B(y)\, R_l(y, x))\, T_l(y, x), \tag{2.4d}$$

$$\frac{\partial T_r}{\partial x}(y, x) = (D(x) + C(x)\, R_r(y, x))\, T_r(y, x), \tag{2.4e}$$

$$\begin{gathered} R_r(\xi, \xi) = R_l(\xi, \xi) = 0, \qquad T_r(\xi, \xi) = T_l(\xi, \xi) = 1, \\ Y \leqslant \xi \leqslant X. \end{gathered} \tag{2.4f}$$

While the differentiations indicated are all with respect to x and y, it should be clear that they can also be with respect to z. Thus, for example, (2.4a) can be written

$$\begin{gathered} \frac{\partial R_r}{\partial z}(y, z) = B(z) + (A(z) + D(z))\, R_r(y, z) + C(z)\, R_r^2(y, z); \\ Y \leqslant y < z \leqslant X; \end{gathered} \tag{2.4a$'$}$$

(2.4c) similarly becomes

$$\frac{\partial T_r}{\partial z}(z, x) = (D(z) + B(z)\, R_r(z, x))\, T_r(z, x), \qquad \text{etc.} \tag{2.4c$'$}$$

Let us now consider a gross approximation to $R_r(y, z)$. We take it to be identically zero. This is compatible when $z = y$ with the condition (2.4f). Physically, we are ignoring all internal reflections. Thus particles are transmitted to the left, but are not allowed to reflect. Equivalently, we are taking an approximate solution to (2.1a, *b*) to be $u(z) \equiv 0$, $v(z) = v_0(z)$, the subscript zero indicating that no reflections are allowed.

Under this approximation we obtain from (2.4c$'$) an approximate T_r function, call it $T_r^{(0)}$:

$$\frac{\partial T_r^{(0)}}{\partial z}(z, x) = D(z)\, T_r^{(0)}(z, x). \tag{2.5}$$

Thus, requiring that the condition (2.4f) hold for $T_r^{(0)}$,

$$T_r^{(0)}(y, x) = \exp\left[\int_y^x D(t)\, dt\right] = v_0(y). \tag{2.6}$$

More generally,

$$T_r^{(0)}(z, x) = \exp\left[\int_z^x D(t)\, dt\right] = v_0(z). \tag{2.7}$$

Actually, as we shall see in Section 6, Eq. (2.7) is the abstract equivalent of the classical WKB approximation used in wave theory and quantum mechanics. In that context, it has long been known (see, for example, [3, 8]) that the WKB wave results when one ignores all internal reflections in the propagating medium. We have done just that in our more general model.

3. The Reflection and Transmission Function Approach to the Bremmer Series

About 25 years ago, H. Bremmer [8] observed that it should be possible in wave propagation problems to start with the WKB wave, keep very careful account of the subsequent multiple internal reflections that it suffers in the propagating medium, and obtain an exact solution to the wave equation. He accomplished this by detailed "counting" arguments, obtaining a formal infinite series whose convergence properties were later studied intensively [1, 5]. We shall now obtain analogous series for the problem (2.1a, b).

We begin by multiplying (2.4e) by $R_r(y, z)$, using (2.4a), and replacing x by z' to get

$$\frac{\partial R_r}{\partial z'}(y, z') = B(z') + A(z')\, R_r(y, z') + R_r(y, z')\, \frac{\partial T_r}{\partial z'}(y, z')/T_r(y, z'). \tag{3.1}$$

(It is easy to see that neither T_r or T_l can be zero.) We next replace T_r by the approximation $T_r^{(0)}$ and define $R_r^{(1)}$ by

$$\begin{aligned} \frac{\partial R_r^{(1)}}{\partial z'}(y, z') &= B(z') + A(z')\, R_r^{(1)}(y, z') \\ &\quad + R_r^{(1)}(y, z')\, \frac{\partial T_r^{(0)}}{\partial z'}(y, z')/T_r^{(0)}(y, z'), \\ R_r^{(1)}(z', z') &= 0, \qquad Y \leqslant z' \leqslant X. \end{aligned} \tag{3.2}$$

Thus,

$$R_r^{(1)}(y, z) = \int_y^z \frac{T_r^{(0)}(y, z)}{T_r^{(0)}(y, z')}\, B(z') \exp\left[-\int_z^{z'} A(t)\, dt\right] dz'. \tag{3.3}$$

While $R_r^{(1)}$ may be considered purely mathematical, it again has significance in the physical model. The function $T_r^{(0)}$ is the transmission function for particles which have experienced no reflections, or, equivalently, changes in direction. These are particles in the zero state. The function $R_r^{(1)}$ is the reflection function for once-reflected particles, those which are in the first state. The reader unfamiliar with the concept of "state" and the corresponding hierarchy of R and T functions will find more detailed information in [6, 12].

All of these ideas are illuminated somewhat if we note that

$$\begin{aligned} v_0(z)\, T_r^{(0)}(y, z) &= \exp\left[\int_z^x D(t)\, dt\right] \exp\left[\int_y^z D(t)\, dt\right] = v_0(y) \\ &= v_0(z')\, T_r^{(0)}(y, z'), \end{aligned} \tag{3.4}$$

so that (3.3) may be written

$$R_r^{(1)}(y, z) = \int_y^z \frac{v_0(z')}{v_0(z)} B(z') \exp\left[-\int_z^{z'} A(t)\, dt\right] dz'. \tag{3.5}$$

Now all reflection and transmission functions are defined in terms of unit input. Thus, $R_r^{(1)}(y, z)$ is the reflection function for once-reflected particles when the value of v at z is unity (see (2.1b) and (2.2)). Because of the structure of (2.1a, b) we may write

$$v_0(z)\, R_r^{(1)}(y, z) = u_1(z). \tag{3.6}$$

The function $u_1(z)$ represents the right-moving flux of once-reflected particles (those in the first state). Equation (3.5) becomes

$$u_1(z) = \int_y^z v_0(z')\, B(z') \exp\left[-\int_z^{z'} A(t)\, dt\right] dz'. \tag{3.7}$$

It is interesting and valuable to notice that a direct analysis of (3.1) with *no* approximations leads to

$$u(z) = \int_y^z v(z')\, B(z') \exp\left[-\int_z^{z'} A(t)\, dt\right] dz'. \tag{3.8}$$

(We omit the details until Section 4.) Thus (3.7) clearly represents the beginning of an iteration scheme.

To proceed further, we multiply (2.4d) by $R_l(y, x)$, use (2.4b), replace y by z' and obtain

$$-\frac{\partial R_l}{\partial z'}(z', x) = C(z') + D(z')\, R_l(z', x) - R_l(z', x) \frac{\partial T_l}{\partial z'}(z', x)/T_l(z', x). \tag{3.9}$$

There is now a strong temptation to index as in (3.2), replacing T_l by $T_l^{(1)}$ and R_l by $R_l^{(2)}$. Unfortunately, $T_l^{(1)}$ has not been defined analytically. To agree with previous ideas, however, $T_l^{(1)}(z', x)$ should be the transmission function for once-reflected particles, giving the flux of such particles at x if the flux is unity at z'. Now the once-reflected particles are described by u_1.

Moreover, no once-reflected particles (indeed, *no* reflected particles at all) can possibly enter the system at $z = x$. Thus, so far as these particles are concerned, we have a problem of the form (2.1a, c), except, of course, that the input at z' is not unity but is rather $u_1(z')$. Hence,

$$u_1(z')\, T_l^{(1)}(z', x) = u_1(x) = u_1(z)\, T_l^{(1)}(z, x). \tag{3.10}$$

The appropriate form of (3.9) is thus

$$-\frac{\partial R_l^{(2)}}{\partial z'}(z', x) = C(z') + D(z')\, R_l^{(2)}(z', x) - R_l^{(2)}(z', x)\, u_1(z)/u_1(z'), \tag{3.11}$$

which leads to

$$u_1(z)\, R_l^{(2)}(z, x) = \int_z^x u_1(z')\, C(z') \exp\left[\int_z^{z'} D(t)\, dt\right] dz' \tag{3.12}$$

where the obvious condition $R_l^{(2)}(x, x) = 0$ has been imposed.

Clearly $R_l^{(2)}$ is the reflection function for twice-reflected particles. These contribute to the v-flux. Since the input at z is not unity but $u_1(z)$, we have

$$v_2(z) = u_1(z)\, R_l^{(2)}(z, x) = \int_z^x u_1(z')\, C(z') \exp\left[\int_z^{z'} D(t)\, dt\right] dz' \tag{3.13}$$

as the left-moving flux of twice-reflected particles.

We may now repeat the kind of argument used to obtain (3.7) and find

$$u_3(z) = \int_y^z v_2(z')\, B(z') \exp\left[-\int_z^{z'} A(t)\, dt\right] dz' \tag{3.14}$$

and, in general,

$$\begin{aligned} v_{2n}(z) &= \int_z^x u_{2n-1}(z')\, C(z') \exp\left[\int_z^{z'} D(t)\, dt\right] dz', \\ u_{2n+1}(z) &= \int_y^z v_{2n}(z')\, B(z') \exp\left[-\int_z^{z'} A(t)\, dt\right] dz', \qquad n = 1, 2, \ldots, \end{aligned} \tag{3.15}$$

with $v_0(z)$ and $u_1(z)$ given by (2.7) and (3.7), respectively. If our reasoning is correct, then it should be the case that problem (2.1a, b) is solved by

$$u(z) = \sum_{n=0}^{\infty} u_{2n+1}(z), \qquad v(z) = \sum_{n=0}^{\infty} v_{2n}(z). \tag{3.16}$$

Of course, the convergence question immediately arises. However, formal substitution of (2.7), (3.7) and (3.15) and equally formal differentiation verify that this is a most reasonable conjecture. We shall refer to the series in (3.16) as the (generalized) Bremmer series for (2.1a, b).

4. A Brief Digression: Some Other Approaches

The reader might wonder why a more direct approach has not been used in deriving (3.15). We have observed that (3.7) is a rather obvious first iteration of (3.8) and that (3.8) is indeed exact. To verify this assertion, we note that (3.1) may be integrated to give

$$R_r(y, z) = \int_y^z \frac{T_r(y, z)}{T_r(y, z')} B(z') \exp\left[-\int_z^{z'} A(t)\,dt\right] dz'. \tag{4.1}$$

Moreover, by definition and linearity,

$$v(z)\, T_r(y, z) = v(y) = v(z')\, T_r(y, z'), \tag{4.2}$$

so that

$$v(z)\, R_r(y, z) = \int_y^z v(z')\, B(z') \exp\left[-\int_z^{z'} A(t)\,dt\right] dz'. \tag{4.3}$$

Once more using the definition and linearity, we find

$$u(z) = v(z)\, R_r(y, z), \tag{4.4}$$

and this yields (3.8).

The analog of (4.1) is obtainable from (3.9)

$$R_l(z, x) = \int_z^x \frac{T_l(z, x)}{T_l(z', x)} C(z') \exp\left[\int_z^{z'} D(t)\,dt\right] dz'. \tag{4.5}$$

There is now a strong temptation to write

$$u(z)\, T_l(z, x) = u(x) = u(z')\, T_l(z', x), \tag{4.6}$$

and proceed. Unfortunately, Eq. (4.6) is *false*. The problem on the interval (z, x) or (z', x) is neither of the form (2.1a, b) nor (2.1a, c). In fact (see [6]), (4.6) must be replaced by

$$u(z)\, T_l(z, x) + R_r(z, x) = u(x) = u(z')\, T_l(z', x) + R_r(z', x). \tag{4.7}$$

Clearly, no expression as simple as (3.8) can be obtained from (4.5) and (4.7).

The reason for this difficulty is that there is an input at $z = x$. The discussion in Section 3 overcomes this by treating particles which have experienced no reflections separately from particles which have suffered one or more reflections. The input of the latter type of particles is always zero.

A possible device suggests itself. Define

$$\tilde{u}(z) = u(z), \qquad \tilde{v}(z) = v(z) - v_0(z). \tag{4.8}$$

Then, clearly,

$$\frac{d\tilde{u}}{dz} = A(z)\,\tilde{u}(z) + B(z)\,\tilde{v}(z) + B(z)\,v_0(z),$$
$$-\frac{d\tilde{v}}{dz} = C(z)\,\tilde{u}(z) + D(z)\,\tilde{v}(z) + \left[D(z)\,v_0(z) + \frac{dv_0}{dz}(z)\right], \tag{4.9a}$$

$$\tilde{u}(y) = 0, \qquad \tilde{v}(x) = 0. \tag{4.9b}$$

The boundary conditions (4.9b) are now as desired, but (4.9a) has become inhomogeneous. Physically there are internal sources. Such systems are usually more difficult to analyze and to understand than homogeneous ones. We therefore pursue this course no further.

To find another approach we return to (3.15). Upon differentiating we obtain

$$\frac{du_{2n+1}}{dz} = A(z)\,u_{2n+1}(z) + B(z)\,v_{2n}(z), \tag{4.10a}$$

$$-\frac{dv_{2n}}{dz} = C(z)\,u_{2n-1}(z) + D(z)\,v_{2n}(z), \tag{4.10b}$$

$$u_{2n+1}(y) = 0, \qquad v_{2n}(x) = 0, \qquad n = 1, 2, 3, \ldots \tag{4.10c}$$

If we define $u_{-1}(z) = 0$ and take $v_0(x) = 1$, then the system is also meaningful for $n = 0$. This set of equations has a physical interpretation, more apparent if we write a finite difference version of (4.10a).

$$u_{2n+1}(z + \Delta) - u_{2n+1}(z) = A(z)\,\Delta u_{2n+1}(z) + B(z)\,\Delta v_{2n}(z) + o(\Delta). \tag{4.11}$$

The right hand side of (4.11) indicates that the change in the flux of right-moving particles, which occurs between z and $z + \Delta$, is due in part to interactions involving such particles and in part to interactions involving left-moving particles. For small Δ ($\Delta > 0$), only one interaction per particle is possible, up to $o(\Delta)$. Thus only u_{2n+1} appears on the right side of (4.11). If u_{2n-1} occurred, two interactions at least would have to take place to make a contribution. On the other hand, v_{2n} appears since the v-type particles must change direction in order to contribute to the u-flux, and the single interaction allows only left-moving particles which have had $2n$ reflections to contribute. A similar interpretation may be made for (4.10b).

If one accepts this rather crude analysis he can arrive at the Bremmer series by starting with the system (4.10). Indeed, Sluijter [11] suggests doing exactly that in the kinds of problems of interest to him. We prefer the treatment of Sections 2 and 3 since it is more in keeping with classical arguments, having at the same time the advantage of eliminating tedious and messy "counting" procedures.

Finally, it is interesting to note that if one does start with (4.10) he can derive equations for the $R^{(n)}$ and $T^{(n)}$ functions of the earlier sections by use of the "state" concept and the methods outlined in [6, 12]. These equations agree with the ones we have already found. However, they are not required in any of the work which follows.

5. The Bremmer Series as a Neumann Series

It was Atkinson [1] who resolved the convergence problem for the classical Bremmer series by noting that it could be viewed as a Neuman series for an appropriate integral equation. We shall use the same idea, but must observe that in early analyses the medium in which the wave moves was taken to be semiinfinite in extent. In our framework, y in (2.1a, b) would have to be chosen as minus infinity to obtain an analogous model. We shall not so restrict our investigation. However, it will do no harm, and will be notationally advantageous, if we take y to be zero. Henceforth, we do so.

Returning to the basic problem (2.1a, b), we note that

$$\frac{d}{dz}\left\{u(z)\exp\left[-\int_0^z A(t)\,dt\right]\right\} = B(z)\,v(z)\exp\left[-\int_0^z A(t)\,dt\right], \tag{5.1a}$$

$$-\frac{d}{dz}\left\{v(z)\exp\left[-\int_z^x D(t)\,dt\right]\right\} = C(z)\,u(z)\exp\left[-\int_z^x D(t)\,dt\right]. \tag{5.1b}$$

If we define

$$\tilde{u}(z) = u(z)\exp\left[-\int_0^z A(t)\,dt\right], \tag{5.2a}$$

$$\tilde{v}(z) = v(z)\exp\left[-\int_z^x D(t)\,dt\right], \tag{5.2b}$$

then (2.1a, b) becomes

$$\frac{d\tilde{u}}{dz} = E(z)\,\tilde{v}(z), \qquad -\frac{d\tilde{v}(z)}{dz} = F(z)\,\tilde{u}(z), \tag{5.3a}$$

$$\tilde{u}(0) = 0, \qquad \tilde{v}(x) = 1, \tag{5.3b}$$

where

$$E(z) = B(z)\exp\left[-\int_0^z A(t)\,dt\right]\exp\left[\int_z^x D(t)\,dt\right], \tag{5.4a}$$

$$F(z) = C(z)\exp\left[\int_0^z A(t)\,dt\right]\exp\left[-\int_z^x D(t)\,dt\right]. \tag{5.4b}$$

It should be noted that the dependence of E and F on x has been suppressed. Also, $E(z)$ and $F(z)$ vanish, if and only if, $B(z)$ and $C(z)$ vanish, respectively.

Equation (5.3) is more convenient for our purposes than (2.1a, b) and we shall henceforth deal with it. For notational convenience we shall also drop the tilde on the u and v.

We now choose to convert our problem to one in integral equation form. Clearly,

$$u(z) = \int_0^z v(t)\, E(t)\, dt, \tag{5.5a}$$

$$v(z) = 1 + \int_z^x u(t) F(t)\, dt. \tag{5.5b}$$

Thus

$$\begin{aligned} u(z) &= \int_0^z E(t) \left\{1 + \int_t^x u(t') F(t')\, dt'\right\} dt \\ &= \int_0^z E(t)\, dt + \int_0^z F(s)\, u(s)\, ds \int_0^s E(t)\, dt + \int_z^x F(s)\, u(s)\, ds \int_0^z E(t)\, dt, \end{aligned} \tag{5.6}$$

or

$$u(z) = \xi(z) + \int_0^x K_1(s, z)\, u(s)\, ds, \tag{5.7}$$

where

$$\xi(z) = \int_0^z E(t)\, dt, \tag{5.8}$$

and

$$K_1(s, z) = \begin{cases} F(s) \int_0^s E(t)\, dt, & s \leqslant z; \\ F(s) \int_0^z E(t)\, dt, & z \leqslant s. \end{cases} \tag{5.9}$$

Similarly,

$$\begin{aligned} v(z) &= 1 + \int_z^x \left\{\int_0^t v(t')\, E(t')\, dt'\right\} F(t)\, dt \\ &= 1 + \int_0^z E(s)\, v(s)\, ds \int_z^x F(t)\, dt + \int_z^x E(s)\, v(s)\, ds \int_s^x F(t)\, dt, \end{aligned} \tag{5.10}$$

or

$$v(z) = 1 + \int_0^x K_2(s, z)\, v(s)\, ds, \tag{5.11}$$

where

$$K_2(s, z) = \begin{cases} E(s) \int_z^x F(t)\, dt, & s \leqslant z; \\ E(s) \int_s^x F(t)\, dt, & z \leqslant s. \end{cases} \tag{5.12}$$

Now in (3.15), replace A and D by zero, and B and C by E and F, respectively. Thus,

$$\begin{aligned} v_{2n}(z) &= \int_z^x u_{2n-1}(z')\, E(z')\, dz', \\ u_{2n+1}(z) &= \int_0^z v_{2n}(z') F(z')\, dz', \qquad n = 1, 2, 3,\ldots \end{aligned} \tag{5.13}$$

From (2.7) and (3.7),

$$v_0(z) = 1, \qquad u_1(z) = \int_0^z E(z')\, dz' = \xi(z). \tag{5.14}$$

It is now clear that the formal series $\sum_{n=0}^{\infty} v_{2n}(z)$ is just the Neumann expansion for the solution of (5.11), while $\sum_{n=0}^{\infty} u_{2n+1}(z)$ is the same kind of expansion associated with (5.7). Henceforth, we shall concentrate mainly on (5.11).

It is a classical result that the Neumann series will converge to the solution of (5.11) provided the first eigenvalue of the homogeneous problem

$$\psi(z) = \lambda \int_0^x K_2(s, z)\, \psi(s)\, ds \tag{5.15}$$

exceeds unity in absolute value. Let λ_1 be this eigenvalue and let ψ_1 be the corresponding eigenfunction. For the moment assume that F is never zero in $[0, x]$. Define (see [11])

$$\eta(s) = \int_s^x |F(t)|\, dt = p, \tag{5.16}$$

and let $\tilde{\eta}$ be the function inverse to η. Also set $q = \eta(z)$, $r = \eta(0)$. Then the inequality

$$\frac{1}{|\lambda_1|} |\psi_1(z)| \leqslant \int_0^x |K_2(s, z)|\, |\psi_1(s)|\, ds, \tag{5.17}$$

becomes

$$\frac{1}{|\lambda_1|} |\psi_1(\tilde{\eta}(q))| \leqslant -\int_r^0 |K_2(\tilde{\eta}(p), \tilde{\eta}(q))|\, |\psi_1(\tilde{\eta}(p))| \frac{dp}{|F(\tilde{\eta}(p))|}. \tag{5.18}$$

Setting $\psi_1(\tilde{\eta}(q)) = \varphi_1(q)$ and using the definition of K_2 then yields

$$\frac{1}{|\lambda_1|} |\varphi_1(q)| \leqslant \int_0^q \frac{|E(\tilde{\eta}(p))|}{|F(\tilde{\eta}(p))|} |\varphi_1(p)| \, p \, dp + \int_q^r \frac{|E(\tilde{\eta}(p))|}{|F(\tilde{\eta}(p))|} |\varphi_1(p)| \, q \, dp$$

$$\leqslant M_2 \left\{ \int_0^q |\varphi_1(p)| \, p \, dp + \int_q^r |\varphi_1(p)| \, q \, dp \right\},$$

where

$$M_2 = \sup_{0 \leqslant p \leqslant r} \frac{|E(\tilde{\eta}(p))|}{|F(\tilde{\eta}(p))|} = \sup_{0 \leqslant z \leqslant x} \frac{|E(z)|}{|F(z)|}. \tag{5.20}$$

We may suppose that

$$\int_0^r |\varphi_1(q)|^2 \, dq = 1. \tag{5.21}$$

Multiplying (5.19) by $|\varphi_1(q)|$ and integrating over $[0, r]$ gives

$$\frac{1}{|\lambda_1|} \leqslant M_2 \left\{ \int_0^r dq \int_0^q |\varphi_1(p)| \, |\varphi_1(q)| \, p \, dp + \int_0^r q \, dq \int_q^r |\varphi_1(p)| \, |\varphi_1(q)| \, dp \right\}. \tag{5.22}$$

Now the right-hand side of (5.22) is exactly $(\tilde{K} |\varphi_1|, |\varphi_1|)$, where

$$\tilde{K}(p, q) = \begin{cases} p, & p \leqslant q; \\ q, & q \leqslant p. \end{cases} \tag{5.23}$$

The eigenvalues $\tilde{\lambda}_i$ of $\tilde{K}$ are well known and, since $\tilde{K}$ is symmetric, it follows that

$$(\tilde{K} |\varphi_1|, |\varphi_1|) \leqslant \frac{1}{|\tilde{\lambda}_1|} = \left(\frac{2r}{\pi}\right)^2. \tag{5.24}$$

Thus we have

$$\frac{1}{|\lambda_1|} \leqslant M_2 \left(\frac{2r}{\pi}\right)^2. \tag{5.25}$$

Recall that the condition for convergence of the Neumann series is $|\lambda_1| > 1$. We therefore are assumed of convergence if

$$r = \eta(0) = \int_0^x |F(t)| \, dt < \frac{\pi}{2(M_2)^{1/2}}. \tag{5.26}$$

THEOREM 1. *The Neumann series (i.e., Bremmer series) for the problem* (5.11) *converges to the solution function v provided F does not vanish and* (5.26) *holds. Similarly, the Neumann series for* (5.6) *converges to the solution function u provided E does not vanish and*

$$\int_0^x |E(t)|\,dt < \frac{\pi}{2(M_1)^{1/2}}, \tag{5.27}$$

where

$$M_1 = \sup_{0\leqslant z\leqslant x} \frac{|F(z)|}{|E(z)|}.$$

Moreover there are problems for which (5.26) *and* (5.27) *cannot be improved.*

Proof. The assertion concerning the v-series has been established. The proof for the u-series proceeds in the same way. We shall not pursue it.

To see that the theorem is "best possible" in the sense stated consider the problem

$$\begin{aligned} \frac{du}{dz} &= kv, \qquad -\frac{dv}{dz} = ku, \\ u(0) &= 0, \qquad v(x) = 1, \end{aligned} \tag{5.28}$$

where k is a positive constant. The solution is trivial:

$$u(z) = \frac{\sin kz}{\cos kx}, \qquad v(z) = \frac{\cos kz}{\cos kx}, \tag{5.29}$$

provided $kx \neq (n + \frac{1}{2})\,\pi$. In particular,

$$u(x) = \tan kx, \qquad v(x) = \sec kx. \tag{5.30}$$

It is easy to verify that the Neumann series for $u(x)$ and $v(x)$ are precisely the power series expansions for tan kx and sec kx. These diverge at $kx = \pi/2$. Since $E = F = k$, $M_1 = M_2 = 1$, conditions (5.26) and (5.27) become

$$\int_0^x k\,dt = kx < \frac{\pi}{2}.$$

This completes the proof.

It is clear that a large number of other estimates for x is available. In particular, estimates involving L_1 and L_2 norms of E and F have been obtained, thus avoiding the strong dependence upon the pointwise behavior of E and F implicit in the definition of M_1 and M_2. However, no other estimate has been found which is "best possible" in the sense of the theorem.

The requirement that F be nonvanishing (in the v-case) is very unpleasant. It is originally made in order that (5.16) defines a function with an inverse.

Obviously at this stage F could be allowed to have zeros at isolated points. However, the value of M_2 is then infinite, in general. Theorem 1 may be somewhat modified to take of certain instances of this kind.

THEOREM 2. *Let $F(t)$ vanish on a set S in $[0, x]$. If $E(t)$ also vanishes on S and we define*

$$M_2^* = \sup_{\substack{0 \leqslant z \leqslant x \\ z \notin S}} \left| \frac{E(z)}{F(z)} \right|, \tag{5.31}$$

then (5.26) *provides a convergence criterion for the v-series provided M_2 is replaced by M_2^*. A similar result holds for the u-series.*

Proof. Define

$$|F_\epsilon(z)| = |F(z)| + \epsilon, \qquad \epsilon > 0,$$

$$M_{2,\epsilon} = \sup_{0 \leqslant z \leqslant x} \frac{|E(z)|}{|F_\epsilon(z)|} = \sup_{\substack{0 \leqslant z \leqslant x \\ z \notin S}} \frac{|E(z)|}{|F_\epsilon(z)|}.$$

Since $|F_\epsilon| > 0$, all the reasoning in the proof of Theorem 1 holds when the $|F|$ in that Theorem is replaced by $|F_\epsilon|$. The convergence criterion becomes

$$\int_0^x |F_\epsilon(t)|\, dt < \frac{\pi}{2(M_{2,\epsilon})^{1/2}}.$$

It is now easy to see that one may let $\epsilon \to 0$. We omit the details.

COROLLARY. *If $|E(z)| = |F(z)|$ on $0 \leqslant z \leqslant x$ then a convergence criterion for both the u and the v-series is*

$$\int_0^x |F(t)|\, dt < \frac{\pi}{2}.$$

Proof. Obvious.

6. SOME APPLICATIONS TO THE WAVE EQUATION

We now turn to the classical area of WKB and Bremmer-type arguments, namely the wave equation in one space dimension.

$$\frac{d^2\psi(z)}{dz^2} + \lambda^2 k^2(z)\, \psi(z) = 0. \tag{6.1}$$

We have written the wave number as λk since in many analyses the behavior of ψ for large λ is of paramount importance.

Our first task is to achieve a "splitting" of (6.1) into a pair of u and v equations. Obviously, this may be accomplished in infinitely many ways. To achieve agreement with some of the work of others, we set

$$\begin{aligned} \psi(z) &= u(z) + v(z), \\ \psi'(z) &= \alpha(z)\, u(z) + \beta(z)\, v(z), \end{aligned} \tag{6.2}$$

where α and β are differentiable, but otherwise arbitrary at present. A bit of labor yields

$$\begin{aligned} \frac{du}{dz} &= \frac{1}{\beta(z) - \alpha(z)} \{[\alpha'(z) + \alpha(z)\,\beta(z) + \lambda^2 k^2(z)]\, u(z) \\ &\quad + [\beta'(z) + \beta^2(z) + \lambda^2 k^2(z)]\, v(z)\}, \\ -\frac{dv}{dz} &= \frac{1}{\beta(z) - \alpha(z)} \{[\alpha'(z) + \alpha^2(z) + \lambda^2 k^2(z)]\, u(z) \\ &\quad + [\beta'(z) + \alpha(z)\,\beta(z) + \lambda^2 k^2(z)]\, v(z)\}. \end{aligned} \tag{6.3}$$

Obviously the additional condition $\beta(z) - \alpha(z) \neq 0$, $0 \leqslant z \leqslant x$, must now be imposed.

We shall analyze (6.3) subject to the conditions

$$u(0) = 0, \qquad v(x) = 1. \tag{6.4}$$

We note in passing that (6.4) produces only one solution to (6.3), and hence to (6.1). In general, a second solution can be obtained by requiring $u(0) = 1$, $v(x) = 0$. For such conditions, the entire analysis of the preceding several sections may be repeated. The complete solution is then obtained by superposition. We prefer not to go into full detail here, leaving such matters to the reader. It should also be apparent from the work of Section 5 that both of these problems are soluble (a condition imposed by Section 2) provided x is so small that all the associated Neumann series converge.

A. *The Classical Case*

Let us return to the mainstream of the discussion and choose, somewhat arbitrarily,

$$\alpha(z) = i\lambda k(z), \qquad \beta(z) = -i\lambda k(z). \tag{6.5}$$

Equation (6.3) becomes, provided $k(z) > 0$ and differentiable

$$\begin{aligned} u'(z) &= \left(i\lambda k(z) - \frac{k'(z)}{2k(z)}\right) u(z) + \frac{k'(z)}{2k(z)}\, v(z), \\ -v'(z) &= \frac{-k'(z)}{2k(z)}\, u(z) + \left(i\lambda k(z) + \frac{k'(z)}{2k(z)}\right) v(z). \end{aligned} \tag{6.6}$$

In the notation of the rest of this paper,

$$A(z) = i\lambda k(z) - \frac{k'(z)}{2k(z)}, \qquad B(z) = \frac{k'(z)}{2k(z)} = -C(k),$$

$$D(z) = i\lambda k(z) + \frac{k'(z)}{2k(z)},$$

$$E(z) = \frac{k'(z)}{2k(z)} \left(\frac{k(x)}{k(0)}\right)^{1/2} \exp\left\{-i\lambda\left[\int_0^z k(t)\,dt - \int_z^x k(t)\,dt\right]\right\}, \tag{6.7}$$

$$F(z) = \frac{k'(z)}{2k(z)} \left(\frac{k(0)}{k(x)}\right)^{1/2} \exp\left\{i\lambda\left[\int_0^z k(t)\,dt - \int_z^x k(t)\,dt\right]\right\},$$

$$M_1 = \frac{k(0)}{k(x)}, \qquad M_2 = \frac{k(x)}{k(0)}.$$

The convergence criterion is

$$\int_0^x \left|\frac{k'(t)}{k(t)}\right| dt < \pi, \tag{6.8}$$

for both the u and the v-series.

The expression for v_0 is

$$\begin{aligned} v_0(z) &= \left(\frac{k(x)}{k(z)}\right)^{1/2} \exp\left[i\lambda \int_z^x k(t)\,dt\right] \\ &= \left(\frac{k(x)}{k(z)}\right)^{1/2} \exp\left[i\lambda\left\{\int_0^x k(t)\,dt - \int_0^z k(t)\,dt\right]\right\}. \end{aligned} \tag{6.9}$$

This is easily seen to be equivalent to the classical WKB solution of the wave equation. The approximation obtained is for a left-moving wave, a result of the boundary conditions (6.4).

It is now easy to see that the u- and v-series are precisely those obtained by Bremmer, save that in the usual treatment the WKB wave moves to the right and the propagating medium is semiinfinite in extent. In that geometry, the convergence condition becomes

$$\int_0^\infty \left|\frac{k'(t)}{k(t)}\right| dt < \pi. \tag{6.10}$$

Atkinson [1] has shown that convergence prevails even with equality in (6.10); this result is beyond the reach of our method.

While the choice of α and β has been described as arbitrary, the coefficients $A(z)$, $B(z)$, etc. which have resulted are physically meaningful. Equations (6.3) are limiting forms of those which occur if the propagating medium is approximated by a layered one of thin lamina, each with its own constant wave number. The lamina are ultimately allowed to approach zero thickness. (For details, see, e.g. [8]).

There are two particularly interesting features of the classical Bremmer series when the parameter λ is taken into account. First, the convergence criterion (6.10) is independent of λ. This is obviously advantageous. On the disadvantageous side is the fact that in each term of the series λ occurs only in a complex exponential. While these exponentials are integrated in the process of obtaining the terms u_{2n+1} and v_{2n}, it is by no means clear how these terms behave as functions of λ. Indeed, the crudest estimates obtained by replacing these complex exponentials by unity obscure the λ dependence completely. We next consider the choice of α and β with an eye to improving this situation.

B. *Extensions of the Classical Case*

The structure of the recursion formulas for u_{2n+1} and v_{2n} makes it evident that it is desirable to have B and C small, provided their smallness does not adversely effect the size of A and D. Suppose we try to choose α and β in (6.3) so that $B = C = 0$:

$$\begin{aligned} \alpha'(z) + \alpha^2(z) + \lambda^2 k^2(z) &= 0, \\ \beta'(z) + \beta^2(z) + \lambda^2 k^2(z) &= 0. \end{aligned} \tag{6.11}$$

A little thought reveals that the problem posed by (6.11) is really equivalent to the problem of solving (6.1). Relaxing our ambitions somewhat we try to solve (6.11) approximately by means of a formal series expansion in $1/\lambda$. (Obviously it suffices to deal just with α).

$$\alpha(z) = a_{-1}(z)\,\lambda + a_0(z) + a_1(z)/\lambda + a_2(z)/\lambda^2 + \cdots . \tag{6.12}$$

Routine manipulations now reveal

$$\begin{aligned} a_{-1}(z) &= \pm ik(z), \\ a'_{-1}(z) &= -2a_{-1}(z)\,a_0(z), \\ a_0'(z) &= -(a_0^2(z) + 2a_{-1}(z)\,a_1(z)), \\ a_1'(z) &= -2(a_0(z)\,a_1(z) + a_{-1}(z)\,a_2(z)), \qquad \text{etc.} \end{aligned} \tag{6.13}$$

If we choose $a_{-1}(z) = +ik(z)$, then

$$\begin{aligned} a_0(z) = -\frac{k'(z)}{2k(z)}, \qquad a_1(z) = \frac{a_0'(z) + a_0^2(z)}{-2ik(z)}, \\ a_2(z) = \frac{a_1'(z) + 2a_0(z)\,a_1(z)}{-2ik(z)}, \qquad \text{etc.} \end{aligned} \tag{6.14}$$

Here we assume that k is sufficiently differentiable for our calculations. Of course, the nonvanishing of k is still essential.

A similar expansion for β,

$$\beta(z) = b_{-1}(z)\,\lambda + b_0(z) + b_1(z)/\lambda + b_2(z)/\lambda^2 + \cdots, \tag{6.15}$$

leads to $b_{-1}(z) = \pm ik(z)$. Obviously, if we choose the plus sign, $\alpha(z) \equiv \beta(z)$, which is not allowable. Picking the minus sign gives

$$\begin{aligned} b_0(z) &= a_0(z), \\ b_1(z) &= -a_1(z), \\ b_2(z) &= a_2(z), \qquad \text{etc.} \end{aligned} \tag{6.16}$$

We make no effort to study the convergence of these formal expansions. Instead we immediately consider the results of truncation.

Case 1. Take

$$a_0 = a_1 = a_2 = \cdots = 0.$$

Here

$$\alpha(z) = i\lambda k(z), \qquad \beta(z) = -i\lambda k(z),$$

and this is just the classical situation considered in A.

Case 2. Take

$$a_1 = a_2 = \cdots = 0.$$

Now

$$\alpha(z) = i\lambda k(z) - k'(z)/2k(z), \qquad \beta(z) = -i\lambda k(z) - k'(z)/2k(z).$$

Hence, from (6.3),

$$\begin{aligned} A(z) &= [i\lambda k'(z) + a_0'(z) + a_0^2(z) + 2\lambda^2 k^2(z)]/(-2i\lambda k(z)), \\ B(z) &= [a_0'(z) + a_0^2(z)]/(-2i\lambda k(z)) = -C(z), \\ D(z) &= [-i\lambda k'(z) + a_0'(z) + a_0^2(z) + 2\lambda^2 k^2(z)]/(-2i\lambda k(z)), \\ a_0(z) &= -k'(z)/2k(z). \end{aligned} \tag{6.17}$$

It is clear that B and C now behave like $1/\lambda$. Moreover, the real parts of A and D are still $\pm k'(z)/2k(z)$, just as in Case 1; only the imaginary parts have changed. The convergence criterion is

$$\int_0^x \left| \frac{a_0'(t) + a_0^2(t)}{k(t)} \right| dt < \pi\lambda. \tag{6.18}$$

Thus, not only do the terms of the Bremmer series get small with large λ (indeed, $u_{2n+1} = O(\lambda^{-2n-1})$, $v_{2n} = O(\lambda^{-2n})$), but also the x values for which the series converge increase with λ.

Case 2 is now new. It has been studied for the semiinfinite medium by Sluijter [11] who was lead to it geometrically. He argued that the thin lamina mentioned in the discussion of the WKB approximation need not have constant wave number but can each be selected with a convenient variable wave number. His choice leads to the above result.

It is quite clear that further improvements on the Bremmer series can be made by simply retaining more terms in the expansions for α and β. The details are tedious and we pursue them no further.

7. Some Miscellaneous Observations

(1) In section 6 the requirement that $k(z)$ not vanish has been essential. A general procedure analagous to that described in Part B of that section and holding when k can vanish (or even when k^2 can be negative, a physically interesting case) is most desirable. As yet no such device has been found. For k simply vanishing, a choice of $\alpha(z) = i(\lambda^2 k^2(z) + b^2)^{1/2}$, b a constant, may be used, with $\beta(z) = -\alpha(z)$. For k^2 negative, the selection

$$\alpha(z) = -\beta(z) = i\lambda(k^2(z) + b^2)^{1/2},$$

where $k^2(z) + b^2 > 0$ for $0 \leqslant z \leqslant x$ is of some value. Neither of these choices produces simultaneously a really satisfactory behavior in the convergence criterion and in B and C.

It is quite possible that to handle these problems successfully one must use a different kind of "splitting" of the wave equation. At the moment the question is still open. (For a quite different treatment of the case $k^2 \leqslant 0$, see [2].)

(2) It is interesting to note that our results have significance for the diffusion equation. Consider the problem

$$\frac{\partial^2 T}{\partial z^2}(z, t) = k^2(z)\frac{\partial T}{\partial t}, \tag{7.1a}$$

$$T(z, 0) = 0, \qquad 0 \leqslant z \leqslant x. \tag{7.1b}$$

Let $\theta(z, s)$ be the Laplace transform of T with respect to t.

$$\theta(z, s) = \int_0^\infty e^{-st}T(z, t)\, dt. \tag{7.2}$$

Then,

$$\frac{d^2\theta}{dz^2} + sk^2(z)\,\theta(z, s) = 0. \tag{7.3}$$

Equation (7.3) is just the wave equation (6.1) with $\lambda = (s)^{1/2}$. Any of the methods of the preceding section can now be used to obtain an expansion of θ, and the resulting Bremmer series may then be inverted, at least formally, to obtain an expansion for $T(z, t)$.

Precisely the same ideas can be applied to the time-dependent one-dimensional transport equation (see [6, 12]). None of these areas has been explored in depth. (For a treatment of the time-dependent wave equation, see [7].)

(3) We also call attention to the fact that our results contain the ingredients for obtaining expansions of the solutions to certain inhomogeneous problems. Consider

$$\begin{aligned} u'(z) &= A(z)\,u(z) + B(z)\,v(z) + S^{+}(z), \\ -v'(z) &= C(z)\,u(z) + D(z)\,v(z) + S^{-}(z), \\ u(0) &= 0, \qquad v(x) = 1, \qquad 0 \leqslant z \leqslant x. \end{aligned} \tag{7.4}$$

Manipulations of the sort employed in Section 5 can now be used to convert this system of equations into the form

$$\begin{aligned} u(z) &= S_1(z) + \int_0^x K_1(s, z)\,u(s)\,ds, \\ v(z) &= S_2(z) + \int_0^x K_2(s, z)\,v(s)\,ds. \end{aligned} \tag{7.5}$$

Here K_1 and K_2 are precisely the kernels given by (5.9) and (5.12) and the S_i functions may be obtained explicitly in terms of S^+, S^-, A, B, etc. Since the Bremmer series are just Neumann series they contain the ingredients for the resolvent kernels associated with K_1 and K_2. When these resolvents are known, (7.5) may be handled at once. We pursue this matter no further, save to note that the specific problem of the wave equation with internal sources is studied in much this way in [4].

8. Summary and Remarks

In this paper we have extended the notion of the WKB method and the Bremmer series to quite general systems of two linear differential equations in two unknowns. By using the concepts and equations from the theory of invariant imbedding, we have made these extensions in a natural manner, one which agrees with physical intuition. At the same time, the results given (except in Section 7) have been rigorously established.

Numerous problems are suggested. Our work has been for the case of scalar equations. Extension to matrix equations would seem to be fairly routine. However, this has not been done, and it is possible that the analysis is not as straightforward as it appears. Also, we have confined our study to linear equations. The concept of reflection and transmission functions has meaning even in the nonlinear case. It seems possible that WKB methods and Bremmer series can be developed for classes of nonlinear problems. No attempt has been made to do this, and one should always approach nonlinear behavior with the expectation that some difficulties will arise.

It would be interesting to obtain "sharp" convergence criteria for the series dealt with in this paper which are less dependent upon the pointwise behavior of the functions involved. As noted in Section 5, additional convergence estimates have been found in terms of norms other than the sup-norm, but they fail to be sharp.

The analysis of the wave equation has been dependent upon the "splitting" chosen. Difficulties in treating the case $k = 0$ suggest that different "splittings" may prove valuable. In fact, given any second-order linear differential equation one is confronted with such a vast infinity of possible splittings it is hard to know how to proceed. In the case of the wave equation we have been guided by both physical ideas and the past experience of others. For other equations there may well be no such guides. It is desirable to develop a notion of "best splitting" ("best" relative to some stated criterion or goal) and to find algorithms for achieving this.

Finally, it seems likely that many of the classical series expansions known in the theory of special functions are really "disguised" Bremmer series. Investigation of this matter may very well be tied to the question of "best splitting" mentioned above.

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